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Limit Theorems for Monolayer Ballistic Deposition in the Continuum

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We consider a deposition model in which balls rain down at random towards a 2-dimensional surface, roll downwards over existing adsorbed balls, are adsorbed if they reach the surface, and discarded if not. We prove a spatial law of large numbers and central limit theorem for the ultimate number of balls adsorbed onto a large toroidal surface, and also for the number of balls adsorbed on the restriction to a large region of an infinite surface.

KEY WORDS: Random sequential adsorption; central limit theorem.

1. INTRODUCTION AND STATEMENT OF RESULTS

Random sequential adsorption (RSA) is a mathematical model, incorporating stochastic and geometric elements, for sequential deposition of colloidal particles or proteins onto a surface (or *substrate*); particles arrive at random locations, and each adsorbed particle occupies a region of the substrate which prevents the adsorption of any subsequently arriving particle in an overlapping surface region. Scientific interest is considerable; for a series of surveys, see *Colloids and Surfaces A* **165** (2000), for example Privman,[12] Senger *et al.*,[14] Talbot *et al.*,[17] and Wang.[18] See Evans[5] for a much-cited earlier survey.

For deposition onto a surface of dimension d, there have been many simulation studies, often concerned with the number of particles ultimately adsorbed onto a region of substrate. It is of interest to know whether this satisfies a law of large numbers (LLN, i.e., a thermodynamic limit) and central limit theorem (CLT, i.e., Gaussian fluctuations) as the region becomes large. While previous rigorous mathematical studies were mainly

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restricted to 1 dimension, for general d Penrose[9] proved a LLN for various continuum systems, and proved both LLNs and CLTs for certain lattice systems.[10] Penrose and Yukich[11] proved both LLNs and CLTs for continuum systems with *finite input* where the addition of incoming particles is terminated before saturation occurs. Except in the case $d=1$ (Dvoretzky and Robbins[4]), a CLT for infinite-input continuum RSA remains elusive.

In the present work we prove a LLN and CLT for an infinite-input continuum model related to RSA which has received attention in its own right on grounds of realism, namely, a form of monolayer *ballistic deposition* (BD), representing deposition in the presence of a gravitational field. Each incoming particle occupies a Euclidean ball of radius ρ in $\mathbb{R}^{d+1}$, with $d=2$ or $d=1$; the $(d+1)$-st coordinate represents "height." An incoming particle falls perpendicularly from above towards a substrate represented by the surface $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$, which we identify with the lower-dimensional space $\mathbb{R}^d$, or a sub-region thereof (the *target region*). Its downward motion is vertical until it hits the substrate or one of the particles previously adsorbed. If it contacts a previously deposited particle, then the new particle rolls, following the path of steepest descent until it reaches a stable position. If the new particle reaches the adsorption surface, it is fixed there; otherwise it is removed from the system. For $d=1$, the model dates back to Solomon (ref. 15, p. 129), and the formulation in $d=2$ by Jullien and Meakin[6] has led to considerable renewed interest; see refs. 14 and 17. We state and prove results only for $d=2$; changing to $d=1$ makes things easier.

To avoid having to specify the behaviour of particles near the boundary of the target region, we assume, as in most simulation studies, that the target region is a *torus* with integer dimensions. Given $A \subset \mathbb{Z}^2$ of the form $A=\{m_1,..., m_2\} \times \{m_3,..., m_4\}$ (a *lattice rectangle*), define $\tilde{A} \subset \mathbb{R}^2$ by

$$\tilde{A} = (m_1 - 1, m_2] \times (m_3 - 1, m_4]. \tag{1.1}$$

We focus attention mainly on target regions of this form, and adopt periodic (toroidal) boundary conditions for the rolling mechanism.

Suppose $X_1, X_2, X_3,...$ are independent and uniformly distributed over $\tilde{A}$. These form the random input to the model with target region $\tilde{A}$; the vector X_i represents the position at which the ith incoming ball would end up touching the 2-dimensional substrate if it were to fall un-hindered. Successive balls are adsorbed (with possible displacement due to rolling) or rejected according to the BD mechanism described above, but adopting the toroidal boundary conditions, whereby an adsorbed particle near one edge of the rectangle $\tilde{A}$ can influence the rolling of a particle near the opposite edge. The process terminates when there is no available space left on the substrate large enough to contain a new item (jamming of $\tilde{A}$); that is, when every point of $\tilde{A}$ lies within a distance less than 2ρ, using the toroidal metric, from some point in $\tilde{A}$ that is the location of the point of contact of some previously adsorbed ball.

Let $N(A)$ denote the (random) number of balls adsorbed at the termination time. Our first result is a LLN for $N(A)$ as A becomes large. For any sequence of sets $(A_n)_{n \geqslant 1}$, set

$$\liminf(A_n) := \bigcup_{n \geqslant 1} \bigcap_{m \geqslant n} A_m.$$

For $p \geqslant 1$, let $\xrightarrow{L^p}$ denote convergence in pth moment as $n \to \infty$.

Theorem 1.1. There is a constant $\mu = \mu(\rho) > 0$ such that if $(A_n)_{n \geqslant 1}$ is a sequence of lattice rectangles satisfying $\liminf(A_n) = \mathbb{Z}^d$, then for any $p \in [1, \infty)$,

$$\frac{N(A_n)}{|A_n|} \xrightarrow{L^p} \mu. \tag{1.2}$$

Next, we give an associated CLT. Let $\mathcal{N}(0, \sigma^2)$ denote a normally distributed random variable with mean zero and variance σ^2, if $\sigma > 0$, or a degenerate random variable taking the value 0 with probability 1, if $\sigma = 0$. Let $\xrightarrow{\mathcal{D}}$ denote convergence in distribution.

Theorem 1.2. There is a constant $\sigma_1 = \sigma_1(\rho) > 0$ such that for any sequence $(A_n)_{n \geqslant 1}$ of lattice rectangles with $\liminf(A_n) = \mathbb{Z}^d$, we have as $n \to \infty$ that

$$|A_n|^{-1} \operatorname{Var}(N(A_n)) \to \sigma_1^2 \tag{1.3}$$

and

$$|A_n|^{-1/2} (N(A_n) - \mathbb{E}N(A_n))) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_1^2).$$

Various alternative boundary conditions, other than the toroidal scheme above, are also feasible. Particles could simply roll until they touch the surface $\mathbb{R}^2$ (possibly outside the target region); or any particle that ends up touching the surface outside the target region could be removed; or (as in Solomon's[15] version of this model, generalized to $d = 2$ in Weiner[19]) the boundary itself could cause a deflection of particles (imagine a "wall" around the boundary of the target region). For the LLN, these boundary conditions are not so important, and a result like Theorem 1.1 can be obtained any boundary conditions provided the influence of the boundary has finite range. Moreover, the target regions in the sequence do not need to be rectangular, provided only that they satisfy a condition of vanishing boundary length relative to their area. We do not go into details on such generalizations because of their proximity to results in ref. 9. For the CLT, however, alternative boundary conditions can cause extra difficulties in the proof. We believe these can be overcome in at least some cases of non-toroidal boundary conditions, but have not written out the details.

While toroidal boundary conditions are usually used in simulation studies, they are not so realistic physically. Another way to avoid boundary effects is to take the whole of $\mathbb{R}^2$ as target region. Our next result shows that a stationary point process, loosely speaking the set of locations of adsorbed points for the BD process with target region $\mathbb{R}^2$, exists as a weak limit of point processes arising from bounded target regions. Let $\mathcal{S}$ be the space of locally finite subsets of $\mathbb{R}^2$. For $\zeta \in \mathcal{S}$ and $B \subset \mathbb{R}^2$, let $\zeta(B)$ denote the number of elements of ζ in B (so $\zeta(\cdot)$ is a counting measure). A *point process* on $\mathbb{R}^2$ is a random element ζ of $\mathcal{S}$. For more details, see, for example, refs. 3, 16, and 13.

If ζ and ζ_n ($n \in \mathbb{N}$) are point processes on $\mathbb{R}^2$, we say the sequence ζ_n converges weakly to ζ if the finite-dimensional distributions converge, i.e., if for any finite collection of bounded Borel sets B_i satisfying $\zeta(\partial B_i) = 0$ almost surely, the joint probability distributions of $\zeta_n(B_i)$ converge weakly to those of $\zeta(B_i)$. This is equivalent to various other definitions of weak convergence; see, e.g., Section 9.1 of ref. 3.

Given a lattice rectangle A, let ξ^A be the point process of locations in $\mathbb{R}^2$ of ultimately accepted particles, for the BD model with target region $\tilde{A}$ (which will be a point process in $\mathbb{R}^2$, all of whose points lie in $\tilde{A}$). Our next result concerns weak convergence of the point process ξ^A as A becomes large. As with Theorem 1.1, the result is not sensitive to the toroidal boundary conditions.

Theorem 1.3. There exists a stationary point process ξ, such that if $(A_n)_{n \geqslant 1}$ is any sequence of lattice rectangles with $\liminf_{n \to \infty} A_n = \mathbb{Z}^2$, the sequence of point processes ξ^{A_n} converges weakly to ξ.

Given any bounded region $B \subset \mathbb{R}^2$, the interpretation of $\xi(B)$ is as follows. The variable $\xi^A(B)$ is the number of adsorbed points in B when the target region is $\tilde{A}$. As A becomes large this has a weak limit which is $\xi(B)$. If one now, in turn, makes B large, it is of interest to know if $\xi(B)$ satisfies a CLT, and our final result says that this is indeed the case. We restrict attention to rectangular regions although other shaped regions can also be dealt with (see the proof).

Theorem 1.4. There exists a constant $\sigma_2 = \sigma_2(\rho) > 0$ such that for any sequence $(B_n)_{n \geqslant 1}$ of lattice rectangles with $\liminf(B_n) = \mathbb{Z}^d$, we have as $n \to \infty$ that

$$|B_n|^{-1} \operatorname{Var}(\xi(\tilde{B}_n)) \to \sigma_2^2 \tag{1.4}$$

and

$$|B_n|^{-1/2} (\xi(\tilde{B}_n) - \mathbb{E}\xi(\tilde{B}_n)) \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma_2^2).$$

Other properties of the point process ξ are also of interest. The proof of Theorem 1.4 involves showing that ξ has exponentially decaying correlations, which is of interest in its own right.

Weiner[20] considered an alternative version of the BD model in two (or more) dimensions in which the region of substrate occupied by a particle is a rectilinear square rather than a circle (the "Solomon model"). He claimed to prove a CLT for the number of particles ultimately adsorbed onto a large target region. However, his argument uses assertions from Weiner,[19] which he later retracted (Weiner[21]). It is possible to adapt our methods to yield a CLT for Weiner's "Solomon model," at least in the case of toroidal boundary conditions, partially vindicating Weiner's claims regarding this model (though not the "Renyi model").

2. GEOMETRIC PRELIMINARIES

In notation used throughout this paper, $\mathbf{0}$ denotes the origin $(0, 0)$ of $\mathbb{R}^2$. For $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\|$ denotes the Euclidean norm (modulus) $\sqrt{x_1^2 + x_2^2}$ of x. For $x \in \mathbb{R}^2$ and $R \subset \mathbb{R}^2$, $x + R$ denotes the translated set $\{x + y\colon y \in R\}$. For $r > 0$, define the continuum disk $D(x; r) \subset \mathbb{R}^2$ and the lattice ball $B(x; r) \subset \mathbb{Z}^2$ by

$$D(x; r) = \{y\colon \|y - x\| \leqslant r\}; \qquad B(x; r) = D(x; r) \cap \mathbb{Z}^2.$$

If E is an event in a given probability space let $\mathbf{1}_E$ be the indicator random variable taking the value 1 if E occurs and 0 if not. Finally, for any directed graph, by a *root* of the directed graph we mean a vertex with indegree zero.

We start with two purely geometric results about the mechanics of the BD model with target region given by the infinite surface $\mathbb{R}^2 \times \{0\}$. Each accepted particle lies on the substrate, and so can be represented by the point in $\mathbb{R}^2$ at which it touches the surface. The position of an accepted particle is a translate (or *displacement*) of the location in $\mathbb{R}^2$ above which it originally comes in. The displacement (and also the decision on whether or not to accept) is determined by the initial location at which the particle comes in, and the positions (after displacement) of the previously accepted particles.

Lemma 2.1. With probability 1, no particle receives a displacement of modulus greater than 8ρ.

Proof. Choi *et al.*[2] enumerate the possible fates an incoming ball might undergo. Since these involve at most 4 deflections, in effect they state the result but do not give a complete proof. Therefore we do so here.

Let each particle already accepted be represented by a point in $\mathbb{R}^2$ located at its point of contact with the substrate. For any two such points the inter-point distance r (say) satisfies $r \geqslant 2\rho$.

For a new particle, let it too be represented by a point in $\mathbb{R}^2$, obtained by projecting the position of its center down onto the substrate (imagine looking down on the substrate from above). As it rolls, the point representing the new particle performs a piecewise linear motion in $\mathbb{R}^2$. The first line segment of this motion represents an initial period when the new particle touches a single existing particle, and is of length at most 2ρ. Thereafter, each successive line segment will represent motion while touching two existing particles, and will be along the mediator (perpendicular bisector) between the two points representing those particles. Let d_j denote the distance between the two particles which the new particle touches during the jth step of linear motion, and note that $2\rho \leqslant d_j < 4\rho$.

Each change in direction of this piecewise linear motion in $\mathbb{R}^2$, say from step j to step $j+1$ of linear motion, will occur at the circumcenter of three existing points. If this circumcenter lies inside the triangle with corners at those three points, then the motion comes to a stop and the particle is discarded, according to the BD rules. Therefore for the motion to continue, the circumcenter lies outside this triangle, so that the triangle must have an obtuse angle. The inter-point distance d_{j+1} is the longest edge length of this triangle, while d_j is one of the other two edge lengths. Since the triangle has an obtuse angle, and all three edges are of length at least 2ρ, we obtain

$$d_{j+1}^2 \geqslant d_j^2 + 4\rho^2,$$

and since the edge-lengths d_j must all be at most 4ρ, this means that the sequence (d_j) terminates in at most three steps, in addition to the initial rolling in contact with just a single previous particle. Since each piecewise linear step is of length at most 2ρ, this completes the proof. ∎

Let us say that after k adsorptions, a given point $x \in \mathbb{R}^2$ is *available* for a particle to be adsorbed, if there are no adsorbed particles touching the surface in $D(x; 2\rho)$.

Lemma 2.2. There exist $\varepsilon_0 > 0$, $\varepsilon_1 > 0$ with the following property. Suppose that for some k, after k adsorptions, a given point $x \in \mathbb{R}^2$ is available for a particle to be adsorbed. Then there exists a region of area at least ε_1 such that any incoming particle with location in that region will be adsorbed in a position that makes all points in $D(x; \varepsilon_0)$ unavailable.

Proof. Assume without loss of generality that $\rho = 1/2$ and $x = \mathbf{0}$. Take $\varepsilon_0 < 1/8$. First suppose no adsorbed particle lies within distance $1+\varepsilon_0$ of the origin. Then any particle arriving in the ball $D(\mathbf{0}; \varepsilon_0)$ will be accepted without rolling, and for such a new particle the unit diameter ball centred at that particle covers the ball $D(\mathbf{0}; \varepsilon_0)$.

Next suppose there already exists a particle (at x, say) with $1 \leqslant \|x\| < 1+\varepsilon_0$. A particle arriving in $D(\mathbf{0}; \varepsilon_0) \cap D(x; 1)$ will receive a first deflection and roll, but not very far. This is because its initial distance from x is more than $1-\varepsilon_0$ so it initially rolls at most a distance ε_0 before it reaches the surface or receives a second deflection. If a second deflection takes place, at that instant the new particle then lies on the mediator of two adsorbed points x, y say. The distance between x and y is more than $2(1-\varepsilon_0)$. If a third deflection were to take place it would be at the circumcenter of adsorbed points points x, y, z, say, making a triangle with an obtuse angle. But this is impossible; for example if (y, z) were the longest edge, the cosine rule would give us

$$\|y-z\|^2 \geqslant \|x-z\|^2 + \|x-y\|^2 \geqslant 1 + (4 - 8\varepsilon_0 + 4\varepsilon_0^2),$$

and therefore $\|y-z\| > 2$ and the third deflection does not take place. Therefore after the second deflection the linear motion terminates either with adsorption or rejection. By Pythagoras' theorem, the distance travelled in this last linear motion after the second deflection is at most

$$\sqrt{1-(1-\varepsilon_0)^2} \leqslant (2\varepsilon_0)^{1/2}$$

and therefore the total displacement of the particle before adsorption is at most $\varepsilon_0 + (2\varepsilon_0)^{1/2}$.

Therefore if a particle arrives within a distance ε_0 of the origin, it is adsorbed or rejected at a distance at most $2\varepsilon_0 + (2\varepsilon_0)^{1/2}$ from the origin. Since $\varepsilon_0 < 1/8$, this is at most $3/4$. If adsorbed, it therefore prevents any subsequent adsorption taking place in $D(\mathbf{0}; \varepsilon_0)$. Therefore we are done, unless there is a possibility of rejection for particles arriving in $D(\mathbf{0}; \varepsilon_0)$.

Next, suppose that it is possible for a particle arriving within distance ε_0 of the origin to be rejected. If this happens it will be at the circumcenter of points x, y, z (say) after initial deflection by x and subsequent deflection by y (say). In this case the circumcenter of x, y, z is at a distance less than 1 from each of x, y, z, and *every point inside the triangle* xyz *is unavailable*. In particular, the origin does not lie inside the triangle xyz; however, it does lie within distance $2\varepsilon_0$ of the midpoint of x and y.

Now suppose a particle lands in $D(\mathbf{0}; \varepsilon_0)$ on the same side of the line xy as the origin. In the course of its subsequent rolling it stays on the same side of the line xy as the origin; if not there would be some other line segment between adsorbed centers, other than $\{x, y\}$, of length between $2(1-\varepsilon_0)$ and 2, whose midpoint lies in $D(\mathbf{0}; 2\varepsilon_0)$, which is geometrically impossible provided ε_0 is sufficiently small. At the end of its motion, if it were rejected, that would take place at the circumcenter of points xyz', say, and in that case all points landing in xyz' would be unavailable. However, provided ε_0 is small enough, the origin must lie in xyz', and therefore we would have a contradiction.

It follows that provided ε_0 is small enough, a point landing in $D(\mathbf{0}; \varepsilon_0)$ on the same side of the line xy as the origin will be accepted, in a position that makes the region $D(\mathbf{0}; \varepsilon_0)$ unavailable. The desired conclusion follows, with $\varepsilon_1 = \pi\varepsilon_0^2/2$. ∎

3. PROBABILISTIC PRELIMINARIES

The author[8, 10] has developed general LLNs and CLTs for functionals on the restriction of spatial white noise processes to finite regions of the lattice, as follows. Suppose $(E, \mathscr{E}, P_0)$ is an arbitrary probability space, and $X = (X_x, x \in \mathbb{Z}^2)$ is a family of independent identically distributed random elements of E, each X_x having distribution P_0. Let X' be the process X with the value $X_\mathbf{0}$ at the origin replaced by an independent copy X_* of $X_\mathbf{0}$ (that is, an E-valued variable X_* with distribution P_0, independent of X), but with the values at all other sites the same.

Let $\mathscr{R}$ denote some collection of nonempty finite subsets of $\mathbb{Z}^2$, with $x + B \in \mathscr{R}$ for all $B \in \mathscr{R}$, $x \in \mathbb{Z}^2$.

A *stationary* $\mathscr{R}$*-indexed functional of* X is a family $H = (H(X; A), A \in \mathscr{R})$ of real-valued random variables, with the property that $(X_x, x \in A)$ determines the value of $H(X; A)$ and $H(\tau_y X; y + A) = H(X; A)$ (almost surely) for all $y \in \mathbb{Z}^2$, where $\tau_y X$ is the family of variables $(X_{x-y}, x \in \mathbb{Z}^2)$. Let $\Delta_0(A)$ be the increment $H(X; A) - H(X'; A)$. The functional H *stabilizes on sequences tending to* $\mathbb{Z}^2$ if there exists a random variable $\Delta_0(\infty)$ such that for any $\mathscr{R}$-valued sequence $(A_n)_{n \geqslant 1}$ with $\liminf_{n \to \infty}(A_n) = \mathbb{Z}^d$, the variables $\Delta_0(A_n)$ converge in probability to $\Delta_0(\infty)$.

A *stationary $\mathscr{R}$-indexed summand* is a collection $(Y_z(X; A),\ A \in \mathscr{R},\ z \in A)$ of real-valued random variables with the property that $(X_x, x \in A)$ determines the value of $Y_z(X; A)$, and $Y_{y+z}(\tau_y X; y+A) = Y_z(X; A)$ (almost surely) for all $y \in \mathbb{Z}^d$, $A \in \mathscr{R}$, $z \in A$. The associated *induced stationary $\mathscr{R}$-indexed sum* is given by $H(X; A) = \sum_{z \in A} Y_z(X; A)$, which is a stationary $\mathscr{R}$-indexed functional.

We restrict attention here to the case where $\mathscr{R}$ is the collection, here denoted $\mathscr{B}$, of all lattice rectangles $\{m_1,..., m_2\} \times \{m_3,..., m_4\}$ with $m_2 - m_1 > 20\rho$ and $m_4 - m_3 > 20\rho$. This is different from the class of sets denoted $\mathscr{B}$ in ref. 10; nevertheless, the following general law of large numbers is proved in just the same manner as the first part of Theorem 3.1 of ref. 10.

Lemma 3.1. Suppose $(Y_z(X; A){:}\ A \in \mathscr{B}, z \in A)$ is a stationary $\mathscr{B}$-indexed summand inducing a stationary $\mathscr{B}$-indexed sum $(H(X; A);\ A \in \mathscr{B})$. Suppose that

$$\sup\{\mathbb{E}\,|Y_0(X; A)|{:}\ A \in \mathscr{B}, \mathbf{0} \in A\} < \infty. \tag{3.1}$$

Suppose there exists an integrable random variable $Y_0(X)$ such that $Y_0(X; B_n) \to Y_0(X)$ in L^1 as $n \to \infty$, for any $\mathscr{B}$-valued sequence $(B_n)_{n \geqslant 1}$ with $\liminf(B_n) = \mathbb{Z}^2$. If $(A_n)_{n \geqslant 1}$ is a $\mathscr{B}$-valued sequence with $\liminf(A_n) = \mathbb{Z}^2$, then

$$|A_n|^{-1}\, H(X; A_n) \xrightarrow{L^1} \mathbb{E}Y_0 \qquad \text{as} \quad n \to \infty. \tag{3.2}$$

Let $\mathscr{F}_0$ be the σ-field generated by $(X_y, y \preccurlyeq \mathbf{0})$, where $y \preccurlyeq \mathbf{0}$ means y precedes or equals $\mathbf{0}$ in the lexicographic ordering on $\mathbb{Z}^2$. The following general CLT is a corollary of Theorem 2.1 of ref. 8 (see Remark (iii) thereafter in ref. 8).

Lemma 3.2. Suppose $(H(X; A);\ A \in \mathscr{B})$ is a stationary $\mathscr{B}$-indexed functional of X which stabilizes on sequences tending to $\mathbb{Z}^2$, and for some $\gamma > 2$ satisfies

$$\sup_{A \in \mathscr{B}}\ \mathbb{E}[|\Delta_0(A)|^\gamma] < \infty. \tag{3.3}$$

Suppose that $(A_n)_{n \geqslant 1}$ is a $\mathscr{B}$-valued sequence with $\liminf(A_n) = \mathbb{Z}^d$. Then as $n \to \infty$, $|A_n|^{-1}\operatorname{Var}(H(X; A_n))$ converges to σ^2, and

$$|A_n|^{-1/2}\,(H(X; A_n) - \mathbb{E}H(X; A_n)) \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma^2), \tag{3.4}$$

with $\sigma^2 = \mathbb{E}[(\mathbb{E}[\Delta_0(\infty) \mid \mathscr{F}_0])^2]$.

We also use the following CLT for stationary random fields, from Bolthausen.[1] For $A_1, A_2 \subset \mathbb{Z}^2$, let $d(A_1, A_2) = \inf\{(\|z_1 - z_2\|: z_i \in A_i, i = 1, 2\}$. Let ∂A_1 be the set of $z \in \mathbb{Z}^2 \setminus A_1$ such that $d(A_1, \{z\}) = 1$.

Lemma 3.3 (ref. 1). Suppose $(\psi_x, x \in \mathbb{Z}^2)$ is a real-valued stationary random field. For integers $a_1, a_2, n \geqslant 1$ define

$$\alpha_{a_1, a_2}(n) = \sup\{|P[F_1 \cap F_2] - P[F_1]\, P[F_2]| : F_i \in \sigma(\psi_z : z \in A_i),\\ |A_i| \leqslant a_i, d(A_1, A_2) \geqslant n\}.$$

Suppose $\sum_{m=1}^{\infty} m\alpha_{a_1, a_2}(m) < \infty$ for $a_1 + a_2 \leqslant 4$, and $\alpha_{1,\infty}(m) = o(m^{-2})$, and $E[|\psi_0|^3] < \infty$, and $\sum_{m=1}^{\infty} m\alpha_{1,1}(m)^{1/3} < \infty$.

Then $\tilde{\sigma}^2 := \sum_{z \in \mathbb{Z}^2} \mathrm{Cov}(\psi_0, \psi_z)$ converges absolutely, and if $\tilde{\sigma}^2 > 0$, then for any sequence $(\Gamma_n)_{n \geqslant 1}$ of subsets of $\mathbb{Z}^2$ with $|\partial(\Gamma_n)|/|\Gamma_n| \to 0$ as $n \to \infty$, $|\Gamma_n|^{-1/2} \sum_{z \in \Gamma_n} \psi_z \xrightarrow{\mathscr{D}} \mathscr{N}(0, \tilde{\sigma}^2)$.

4. PROOF OF LLN

Let $\mathscr{P}$ be a homogeneous Poisson point process of unit intensity on $\mathbb{R}^2 \times [0, \infty)$. Given $A \in \mathscr{B}$, label the points of the restriction of $\mathscr{P}$ to $\tilde{A} \times [0, \infty)$ as $\{(X_i(A), T_i(A))\}_{i=1}^{\infty}$ with $T_1(A) < T_2(A) < T_3(A) < \cdots$. Throughout the proofs of Theorems 1.1–1.3, we assume without loss of generality that the random input for the variable $N(A)$, defined in the introduction, is given by the sequence of variables $X_1(A), X_2(A), \ldots$ representing the locations of successive incoming particles (thus $T_i(A)$ is taken to be the time of arrival of the ith incoming particle). By this device, coupled realizations of $N(A)$ are defined for all $A \in \mathscr{B}$ simultaneously.

For each point (X, T) of the restriction of $\mathscr{P}$ to $\tilde{A} \times [0, \infty)$, define the pair $I(X, T; A) = (I_0(X, T; A), I_{\rightarrow}(X, T; A))$, with $I_0(X, T; A) \in \{0, 1\}$ and $I_{\rightarrow}(X, T; A) \in \mathbb{R}^2$, as follows. Let $I_0(X, T; A)$ (an indicator variable) be equal to 1 if the ball arriving at location X at time T is accepted, and to zero if it is rejected, in the realisation of the BD model with target set $\tilde{A}$ described above. If $I_0(X, T; A) = 1$, let $I_{\rightarrow}(X, T; A)$ denote the lateral displacement received by the particle arriving at X at time T, prior to being adsorbed. If $I_0(X, T; A) = 0$, let $I_{\rightarrow}(X, T; A) = \mathbf{0}$.

By Lemma 2.1, the decision on whether to accept an incoming particle, and also its displacement if accepted, are determined by the locations (after displacement) of those particles lying within a distance at most 10ρ from the location at which the new ball arrives.

The proof of the LLN and CLT involves a graphical construction. Make the Poisson process $\mathscr{P}$ on $\mathbb{R}^2 \times [0, \infty)$ into the vertex set of an (infinite) oriented graph, denoted $\mathscr{G}$, by putting in an oriented edge $(X, T) \to (X', T')$ whenever $\|X' - X\| \leqslant 20\rho$ and $T < T'$. Observe that particle (X, T) cannot affect (X', T') directly unless there is an edge $(X, T) \to (X', T')$ or the toroidal boundary conditions come into play.

For $z = (z_1, z_2) \in \mathbb{Z}^2$, and $\varepsilon > 0$, define the squares

$$Q_{z,\varepsilon} := ((z_1 - 1)\,\varepsilon, z_1\varepsilon] \times ((z_2 - 1)\,\varepsilon, z_2\varepsilon]; \qquad Q_z := Q_{z,1}.$$

For $x, y \in \mathbb{Z}^2$, let us say that *y is affected by x before time t* if there exists a (directed) path in the oriented graph that starts at some Poisson point (X, T) with $X \in Q_x$, and ends at some Poisson point (Y, U) with $Y \in Q_y$ and $U \leqslant t$. Let $E_t(x, y)$ denote the event that y is affected by x before time t.

Lemma 4.1. There is a constant $\delta_1 \in (0, 1)$ such that for all $x, y \in \mathbb{Z}^d$,

$$P[E_{\delta_1 \|x-y\|}(x, y)] \leqslant 2(3^{-\|x-y\|}).$$

Proof. See Lemma 3.1 of ref. 9. This applies directly if $20\rho \leqslant 1$, and its proof is easily adapted to the case $20\rho > 1$. ▮

For $z \in \mathbb{Z}^d$ and $t > 0$, define the *cluster* $C_{z,t} \subset \mathbb{Z}^d$ by

$$C_{z,t} := \{x \in \mathbb{Z}^d : z \text{ is affected by } x \text{ before time } t\}, \tag{4.1}$$

which is almost surely finite by Lemma 4.1 and the Borel–Cantelli lemma. Let $B_{z,t}$ be the smallest element of $\mathscr{B}$ that contains $\bigcup_{x \in C_{z,t}} B(x; 4 + 20\rho)$. Note that $B_{z,t}$ includes a "buffer zone" around $C_{z,t}$ so that

$$\operatorname{dist}\left(\bigcup_{y \in \mathbb{Z}^d \setminus B_{z,t}} Q_y, \bigcup_{x \in C_{z,t}} Q_x\right) > 20\rho,$$

so that even if we were to add extra points outside the union of squares Q_y, $y \in B_{z,t}$, there will not be any connected path in the graph from any of these added points to any Poisson point $(X, T) \in Q_z \times (0, t]$. This will be important later on.

Lemma 4.2. Suppose $z \in \mathbb{Z}^2$, $t > 0$. If A is a lattice box with $B_{z,t} \subseteq A$, then for all Poisson points (X, T) lying in $Q_z \times [0, t]$ we have $I(X, T; A) = I(X, T; B_{z,t})$.

Proof. By definition, the influence of Poisson points outside $B_{z,t}$ does not propagate to any Poisson points in $Q_z \times [0, t]$. Therefore the fate of such points is the same whether the target region is $\tilde{A}$ or $\tilde{B}_{z,t}$. ▮

Let $\mathscr{S}_0$ be the space of all finite subsets S of $D(\mathbf{0}; 10\rho)$ such that $\|x-y\| \geqslant 2\rho$ for all distinct $x, y \in S$. Define $\Psi_0: \mathscr{S}_0 \to \{0, 1\}$ and $\Psi_{\rightarrow}: \mathscr{S}_0 \to \mathbb{R}^2$ as follows. For $S \in \mathscr{S}_0$, let $\Psi_0(S)$ take the value 1 (respectively 0) if an incoming particle at the origin of $\mathbb{R}^2$ is accepted (respectively rejected), given that S is the configuration of previously accepted particles in $B(\mathbf{0}; 10\rho)$. If $\Psi_0(S) = 1$, let $\Psi_{\rightarrow}(S) \in \mathbb{R}^2$ be the (lateral) displacement of an incoming particle at the origin of $\mathbb{R}^2$, prior to acceptance, given that $\mathscr{S}$ is the configuration of previously accepted particles in $B(\mathbf{0}; 10\rho)$. If $\Psi_0(S) = 0$, let $\Psi_{\rightarrow}(S) = \mathbf{0} \in \mathbb{R}^2$.

We construct a spatially homogeneous form of the BD model with the whole of $\mathbb{R}^2$ as target region, as follows. Define subsets G_i, $i = 0, 1, 2, 3,\ldots$, of $\mathscr{P}$ as follows. Let G_0 be the set of roots of $\mathscr{G}$, and recursively, if $G_0,\ldots, G_n$ are defined, let G_{n+1} be the set of roots of the graph $\mathscr{G}$ with all vertices in $G_0,\ldots, G_n$ removed. As a consequence of Lemma 4.1, the sets $G_0, G_1, G_2,\ldots$ form a partition of $\mathscr{P}$ (see ref. 9, Lemma 3.2).

Define $I(X, T) = (I_0(X, T), I_{\rightarrow}(X, T))$ with $I_0(X, T) \in \{0, 1\}$ and $I_{\rightarrow}(X, T) \in \mathbb{R}^2$ (representing acceptance status and lateral displacement respectively) for $(X, T) \in G_0, G_1,\ldots$ as follows. If $(X, T) \in G_0$ then set $I_0(X, T) = 1$ and $I_{\rightarrow}(X, T) = \mathbf{0}$. Recursively for $n = 1, 2, 3,\ldots$, for $(X, T) \in G_n$, set

$$S_{X,T} = \Big\{ Y + I_{\rightarrow}(Y, U) - X : (Y, U) \in \bigcup_{m=0}^{n-1} G_m, I_0(Y, U) = 1,$$

$$\|Y + I_{\rightarrow}(Y, U) - X\| \leqslant 10\rho \Big\},$$

then set $I_0(X, T) = \Psi_0(S_{X,T})$ and $I_{\rightarrow}(X, T) = \Psi_{\rightarrow}(S_{X,T})$.

For $t > 0$, let ξ_t be the point process of positions *after displacement* of particles accepted up to time t; that is, re-labelling the points of $\mathscr{P}$ in arbitrary order as $\{(X_j, T_j)\}_{j=1}^{\infty}$, let ξ_t be the random locally finite set in $\mathbb{R}^2$ defined by

$$\xi_t = \{X_j + I_{\rightarrow}(X_j, T_j) : I_0(X_j, T_j) = 1, T_j \leqslant t\}. \tag{4.2}$$

This point process is now rigorously defined in terms of the Poisson process $\mathscr{P}$ and the graphical construction. It is a stationary point process on $\mathbb{R}^2$. Define the limiting point process

$$\xi = \bigcup_{t \geqslant 0} \xi_t. \tag{4.3}$$

Similarly, for $A \in \mathcal{B}$ let ξ_t^A be the set of locations after displacement (using the toroidal boundary conditions) of points (X, T) of $\mathcal{P} \cap (\tilde{A} \times [0, \infty))$ such that $T \leqslant t$ and $I_0(X, T; A) = 1$. All points of ξ_t^A lie in $\tilde{A}$. Define the limiting point process $\xi^A = \bigcup_{t \geqslant 0} \xi_t^A$, a point process in $\tilde{A}$.

Choose $\varepsilon_2 \in (0, \varepsilon_0/2)$, with ε_0 taken from Lemma 2.2, and with $1/\varepsilon_2 \in \mathbb{N}$. For $z \in \mathbb{Z}^d$, let β_z denote the (random) time at which the square Q_{z, ε_2} becomes blocked, i.e., the first time at which the point process ξ_t leaves no part of the surface of the square Q_{z, ε_2} still available to adsorb a sphere. For $A \in \mathcal{B}$, define β_z^A in the same way with respect to the point process ξ_t^A. The next result says that the variables β_z and β_z^A are almost surely finite and in fact their distributions have tails which decay exponentially, uniformly in z, A.

Lemma 4.3. It is the case that

$$\limsup_{t \to \infty} t^{-1} \sup_{z \in \mathbb{Z}^2} \{\log P[\beta_z \geqslant t]\} < 0, \tag{4.4}$$

and

$$\limsup_{t \to \infty} t^{-1} \sup\{\log P[\beta_z^A \geqslant t] \colon A \in \mathcal{B}, z \in \mathbb{Z}^2, Q_{z, \varepsilon_2} \subset \tilde{A}\} < 0. \tag{4.5}$$

Proof. Suppose $\beta_z > t$, i.e., at time t there exists a point $x \in Q_{z, \varepsilon_2}$ that is not yet covered, i.e., still available. Then by Lemma 2.2, the probability of a particle arriving and causing Q_{z, ε_2} to be covered by time $t + h$ is at least $\varepsilon_1 h + o(h)$. Therefore we can choose $h_1 > 0$ such that

$$P[\beta_z > t + h_1 \mid \beta_z > t] \leqslant 1 - \varepsilon_1 h_1/2,$$

so that by induction, $P[\beta_z > n h_1] \leqslant (1 - \varepsilon_1 h_1/2)^n$ for all $n \in \mathbb{N}$. This argument holds uniformly in z, and (4.4) follows. Furthermore, the same argument carries through to the torus, to give (4.5). ▮

For $y \in \mathbb{Z}^d$, let

$$J_y = \max\{\beta_z \colon Q_{z, \varepsilon_2} \subset Q_x, x \in B(y; 4 + 20\rho)\}, \tag{4.6}$$

and (for $y \in A \in \mathcal{B}$, with $\|\cdot\|_A$ denoting the toroidal metric)

$$J_y^A = \max\{\beta_z^A \colon Q_{z, \varepsilon_2} \subset Q_x, x \in A, \|x - y\|_A \leqslant 4 + 20\rho\}. \tag{4.7}$$

For each $x \in B(y; 4+20\rho)$, the square Q_x is jammed by the point process ξ_{J_y}, meaning that it is not possible for any Poisson point arriving after time J_y to be accepted at a position in Q_x. In particular, by Lemma 2.1, all particles arriving in Q_y after time J_y are rejected. Define the enlarged "cluster" B'_y by

$$B'_y = \bigcup_{x \in B(y; 4+20\rho)} B_{x, J_y}.$$

Using this enlarged cluster we can strengthen Lemma 4.2 to account for arrivals at *all* times, as follows.

Lemma 4.4. Suppose $y \in \mathbb{Z}^2$. If A is a lattice rectangle with $B'_y \subseteq A$, then for all Poisson points (X, T) lying in $Q_y \times [0, \infty)$ we have $I(X, T; A) = I(X, T; B'_y) = I(X, T)$.

Proof. Suppose (X, T) is a Poisson point in $Q_x \times [0, J_y]$, for some $x \in B(y; 4+20\rho)$. Then by Lemma 4.2, we have $I(X, T; A) = I(X, T; B'_y) = I(X, T)$.

By definition of J_y, it follows that the restriction of the point process $\xi^A_{J_y}$ to the set $\bigcup_{x \in B(y; 2+10\rho)} Q_x$ precludes the subsequent adsorption of any more particles in $\bigcup_{x \in B(y; 2+10\rho)} Q_x$, and in particular prevents acceptance of any subsequent particles arriving in Q_y; therefore for every Poisson point $(X, T) \in Q_y \times (J_y, \infty)$, we have $I(X, T; A) = I(X, T; B'_y) = (0, \mathbf{0})$. ∎

By Lemma 4.3, the variable J_y has an exponentially decaying tail, uniformly in y, i.e.,

$$\limsup_{t \to \infty} t^{-1} \sup_{y \in \mathbb{Z}^d} \log P[J_y > t] < 0. \tag{4.8}$$

For $z \in \mathbb{Z}^d$, let $\mathscr{X}_z$ be the image of the restriction of $\mathscr{P}$ to $Q_z \times [0, \infty)$, under the translation that sends each point (X, T) to $(X - z, t)$. This is a homogeneous Poisson point process on $Q_0 \times [0, \infty)$. The Poisson processes $(\mathscr{X}_z, z \in \mathbb{Z}^2)$ are independent identically distributed random elements of a measurable space $(E, \mathscr{E})$, where E is the space of locally finite subsets of $Q_0 \times [0, \infty)$. The idea behind the proof of Theorems 1.1 and 1.2 is to regard $N(A)$ as a stationary $\mathscr{B}$-indexed functional driven by the process $\mathscr{X} = (\mathscr{X}_z)_{z \in \mathbb{Z}^2}$, and use Lemmas 3.1 and 3.2 from Section 3. We write $\mathscr{X}$ rather than X in this case.

Proof of Theorem 1.1. For $A \in \mathscr{B}$ and $z \in \mathbb{Z}^2$, set

$$Y_z(\mathscr{X}; A) = \sum_{(X, T) \in \mathscr{P} \cap (Q_z \times [0, \infty))} I(X, T; A). \tag{4.9}$$

Then $(Y_z(\mathscr{X}; A), A \in \mathscr{B}, z \in A)$ defined by (4.9) is a stationary $\mathscr{B}$-indexed summand on on the process $\mathscr{X} = (\mathscr{X}_z)_{z \in \mathbb{Z}^2}$, and the corresponding stationary $\mathscr{B}$-indexed sum $H(\mathscr{X}; A)$ is equal to $N(A)$. It suffices to check the conditions in the general result Lemma 3.1. Since the variables $Y_z(\mathscr{X}; A)$ are uniformly bounded by a constant, (3.1) holds.

Suppose $(A_n)_{n \geqslant 1}$ is a $\mathscr{B}$-valued sequence with $\liminf(A_n) = \mathbb{Z}^d$. Then there exists a random variable N_1 such that for all $n \geqslant N_1$, $B'_0 \subseteq A_n$. By Lemma 4.4, for all $n \geqslant N_1$ and all Poisson points in $Q_0 \times [0, \infty)$ we have $I(X, T; A_n) = I(X, T; B'_0)$. Hence $Y_0(\mathscr{X}; A_n) = Y_0(\mathscr{X}; B'_0)$ for $n \geqslant N_1$, so $Y_0(\mathscr{X}; A_n)$ converges almost surely to $Y_0(\mathscr{X}; B'_0)$ as $n \to \infty$. Therefore Lemma 3.1 applies to give us (1.2) with convergence in L^1. Convergence in L^p then follows since $N(A)/|A|$ is uniformly bounded by a constant. ∎

Proof of Theorem 1.3. Suppose $(A_n)_{n \geqslant 1}$ is a $\mathscr{B}$-valued sequence with $\liminf(A_n) = \mathbb{Z}^2$. It suffices to prove that for any bounded Borel $B \subset \mathbb{R}^d$ we have *almost sure* convergence

$$\xi^{A_n}(B) \to \xi(B). \tag{4.10}$$

If n is sufficiently large so that $B'_z \subseteq A_n$ for all z within distance $4 + 10\rho$ of B, then by Lemma 4.4, $\xi^{A_n}(B) = \xi(B)$, which gives us (4.10). ∎

5. PROOF OF CLTS

With $\mathscr{X}$ as defined in the previous section, $(N(B), B \in \mathscr{B})$ is a stationary $\mathscr{B}$-indexed functional driven by the white noise process $\mathscr{X}$. Our goal is to apply Lemma 3.2.

In this setting, the process $\mathscr{X}'$ appearing in the conditions for Lemma 3.2 is obtained from the process $\mathscr{P}'$ defined as follows. Let $\mathscr{X}'$ be the Poisson process obtained by replacing the restriction $\mathscr{X}_0$ of $\mathscr{P}$ to $Q_0 \times [0, \infty)$ with an independent Poisson process $\mathscr{X}_*$ on $Q_0 \times [0, \infty)$, so that

$$\mathscr{X}' = (\mathscr{P} \setminus (Q_0 \times [0, \infty))) \cup \mathscr{X}_*. \tag{5.1}$$

The points of $\mathscr{P}' \setminus (Q_0 \times [0, \infty))$ are the same as those of $\mathscr{P} \setminus (Q_0 \times [0, \infty))$. However, the decision on whether to accept may be different; let $I'(X, T) = (I'_0(X, T), I'_{\to}(X, T))$ be defined in the same manner as $I(X, T)$ but based on the process generated by $\mathscr{P}'$ rather than $\mathscr{P}$; likewise, given $A \in \mathscr{B}$, for $(X, T) \in \mathscr{P}' \cap (\tilde{A} \times [0, \infty))$ let $I'(X, T; A) = (I'_0(X, T; A), I'_{\to}(X, T; A))$ be defined in the same manner as $I(X, T; A)$ but based on the process generated by $\mathscr{P}'$ rather than $\mathscr{P}$.

Lemma 5.1. Let $z \in \mathbb{Z}^d$, and $t > 0$. Suppose $\mathbf{0} \notin B'_z$. Then $I(X, T; A) = I'(X, T; A)$ for all points (X, T) of $\mathscr{P}$ in $Q_z \times [0, \infty)$, and all $A \in \mathscr{B}$ with $B'_z \subset A$.

Proof. Suppose (X, T) is a point of $\mathscr{P}$, with $X \in Q_z$. Then

$$I(X, T; A) = I(X, T; B'_z) = I'(X, T; B'_z) = I'(X, T; A),$$

where the first equality comes from Lemma 4.4, and the second comes from the equality of $\mathscr{P}$ and $\mathscr{P}'$ outside $Q_0 \times [0, \infty)$. ▮

The idea for proving stabilization goes as follows. By Lemma 4.1, the effect of changing the inputs at the origin propagates like an "infection" spreading through space at a linear rate. However, if this "infection" encounters a "wall" of thickness 10ρ surrounding the origin, consisting of sites which are entirely blocked before the infection reaches them, then this wall prevents any spread of the effect of changing inputs at the origin to the other side of the wall. The existence with high probability of a wall surrounding the origin follows from the fact that the probability that a site is not blocked by time t decays exponentially in t.

By Lemma 4.1 and (4.8), $P[\mathbf{0} \in B'_z]$ decays exponentially in $\|z\|$. Therefore, by the Borel–Cantelli lemma,

$$P[\mathbf{0} \in B'_z \text{ for infinitely many } z] = 0. \tag{5.2}$$

For $t > 0$, define the annular region

$$\mathscr{A}_t := \bigcup_{z \in B(\mathbf{0};\, t+4+20\rho) \setminus B(\mathbf{0};\, t)} Q_z$$

and the "distant" set

$$\mathscr{D}_t := \bigcup_{z \in \mathbb{Z}^2 \setminus B(\mathbf{0};\, t+4+20\rho)} Q_z.$$

Lemma 5.2. Let $t > 0$, and suppose $\mathbf{0} \notin B'_z$ for all $z \in \mathscr{A}_t \cap \mathbb{Z}^d$. Then $I(X, T; A) = I'(X, T; A)$ for all points (X, T) of $\mathscr{P}$ in $\mathscr{A}_t \cup \mathscr{D}_t$, and all $A \in \mathscr{B}$ with $B'_z \subset A$ for all $z \in \mathscr{A}_t \cap \mathbb{Z}^d$.

Proof. Suppose $\mathbf{0} \notin B'_z$ for all $z \in \mathscr{A}_t$. Suppose also $A \in \mathscr{B}$ with $B'_z \subset A$ for all $z \in \mathscr{A}_t$. By Lemma 5.1,

$$I(X, T; A) = I'(X, T; A), \qquad \forall (X, T) \in \mathscr{P} \cap (\mathscr{A}_t \times [0, \infty)). \tag{5.3}$$

Next consider Poisson points (X, T) with $X \in \mathcal{D}_t$. Any path to (X, T) from $Q_0 \times [0, \infty)$ must pass through $\mathcal{A}_t \times [0, \infty)$, and the status of all points in this region is unaffected by the change in $\mathcal{P}_0$, so that $I(X, T; A) = I'(X, T; A)$. More formally, we use an induction, as follows.

Define generations $G_0(A, t), G_1(A, t), \ldots$ as follows. Let $\mathcal{G}(A, t)$ be the restriction of $\mathcal{G}$ to vertices in $(\mathcal{D}_t \cap \tilde{A}) \times [0, \infty)$. Let $G_0(A, t)$ be the set of roots of $\mathcal{G}(A, t)$. Then remove vertices of $G_0(A, t)$ from $\mathcal{G}(A, t)$ and let $G_1(A, t)$ be the set of roots of the remaining oriented graph. Then remove vertices in $G_1(A, t)$ too and let $G_2(A, t)$ be the set of roots of the remaining graph and so on. The sets $G_0(A, t), G_1(A, t), \ldots$ form a partition of the vertex set of $\mathcal{G}(A, t)$, because $C_{z,t}$ defined at (4.1) is finite for all z, t.

The inductive hypothesis is that if $(X, T) \in G_n(A, t)$, then $I(X, T; A) = I'(X, T; A)$. This is true for $n = 0$, because if $(X, T) \in G_0(A, t)$, then any (X', T') for which $X' \in \tilde{A}$ and there is an edge from (X', T') to (X, T) lies in the annulus $\mathcal{A}_t \times [0, \infty)$, and therefore by (5.3) satisfies $I'(X', T'; A) = I(X', T'; A)$, which implies $I'(X, T; A) = I(X, T; A)$, since the decision on the value of $I(X, T; A)$ depends only on the decisions at points (X', T') from which there are edges to (X, T)

Now suppose the hypothesis is true for $n = 0, 1, \ldots, k-1$. Then if $(X, T) \in G_k(A, t)$ all of the points (X', T') from which there is an edge to (X, T) lie either in one of the generations $G_0(A, t), \ldots, G_{k-1}(A, t)$, or in $\mathcal{A}_t \times [0, \infty)$, and therefore, by the inductive hypothesis and by (5.3), all such (z', T') satisfy $I(z', T'; A) = I'(z', T'; A)$, and hence again $I'(z, T; A) = I(z, T; A)$. ∎

Lemma 5.3. If for $A \in \mathcal{B}$ we set

$$\Delta_0(A) = \sum_{z \in A} \sum_{(X, T) \in \mathcal{P}:\, X \in Q_z} (I(X, T; A) - I'(X, T; A))$$

then

$$\sup_{A \in \mathcal{B}} \mathbb{E}[|\Delta_0(A)|^3] < \infty. \tag{5.4}$$

Proof. Modify the graphical construction from Section 4 to produce an oriented graph $\mathcal{G}^A$ with vertex set the Poisson process $\mathcal{P} \cap (\tilde{A} \times [0, \infty))$, by putting in an oriented edge $(X, T) \to (X', T')$ whenever $\|X' - X\|_A \leqslant 20\rho$ and $T < T'$, where $\|\cdot\|_A$ denotes the toroidal metric. For $x, y \in A$, let us say that *y is affected in A by x before time t* if there exists a (directed) path in the oriented graph $\mathcal{G}_A$ that starts at some Poisson point (X, T) with $X \in Q_x$, and ends at some Poisson point (Y, U) with $Y \in Q_y$ and $U \leqslant t$. For

$z \in A$ and $t > 0$, define the "A-cluster" C_z^A to consist of all $x \in A$ such that some $y \in A$ with $\|y - z\|_A \leqslant 4 + 20\rho$ is affected in A by x before time J_y^A.

A similar argument to the proof of Lemma 5.1 yields $I(X, T; A) = I'(X, T; A)$ for all (X, T) with $X \in Q_z$ and $B(\mathbf{0}; 4 + 20\rho) \cap C_z^A = \varnothing$. Hence there is a constant c such that for any $A \in \mathscr{B}$,

$$\Delta_0(A) \leqslant c \sum_{z \in A} \mathbf{1}_{\{B(\mathbf{0};\, 4+20\rho) \cap C_z^A \neq \varnothing\}}.$$

However, $P[B(\mathbf{0}; 4 + 20\rho) \cap C_z^A \neq \varnothing]$ decays exponentially in $\|z\|_A$, uniformly over $A \in \mathscr{B}$, because of Lemmas 4.1 and 4.3. The bounded moments condition (5.4) follows. ▮

Proof of Theorem 1.2. As in the proof of Theorem 1.1, define $Y_z(\mathscr{X}; A)$ by (4.9). The aim is to apply Lemma 3.2. The bounded moments condition (3.3) follows from Lemma 5.3.

We need to check stabilization. By (5.2), there exists a random R such that $0 \notin B'_z$ for $z \in \mathbb{Z}^d$ with $\|z\| \geqslant R$. Let B_∞ be the smallest element of $\mathscr{B}$ containing $\bigcup_{z \in B(\mathbf{0};\, R)} B'_z$. Suppose $(A_n)_{n \geqslant 1}$ is a $\mathscr{B}$-valued sequence with $\liminf(A_n) = \mathbb{Z}^d$. Then there exists random N_2 such that

$$B_\infty \subset A_n, \qquad \text{for all} \quad n \geqslant N_2.$$

and such that a similar expression holds with regard to the Poisson process $\mathscr{P}'$ rather than $\mathscr{P}$. Then by Lemma 4.4, for all $n \geqslant N_2$ and for all $z \in B(\mathbf{0}; R)$, we have

$$Y_z(\mathscr{X}; A_n) - Y_z(\mathscr{X}'; A_n) = Y_z(\mathscr{X}; B_\infty) - Y_z(\mathscr{X}'; B_\infty) \qquad \text{for all} \quad n \geqslant N_2. \tag{5.5}$$

Also, there exists random $N_3 \geqslant N_2$ such that for all $n \geqslant N_3$ we have

$$B'_z \subset A_n \qquad \text{for all} \quad z \in \mathscr{A}_R \cap \mathbb{Z}^d.$$

Hence by the definition of R and Lemma 5.2, for all $z \in \mathbb{Z}^d \setminus B(\mathbf{0}; R)$,

$$Y_z(\mathscr{X}; A_n) = Y_z(\mathscr{X}'; A_n) \qquad \text{for all} \quad n \geqslant N_3.$$

Combined with (5.5), this gives us for all $n \geqslant N_3$,

$$\sum_{z \in A_n} (Y_z(\mathscr{X}; A_n) - Y_z(\mathscr{X}'; A_n)) = \sum_{z \in B(\mathbf{0};\, R)} (Y_z(\mathscr{X}; B_\infty) - Y_z(\mathscr{X}'; B_\infty)),$$

which demonstrates stabilization of the induced functional

$$H(\mathscr{X}; A) = \sum_{y \in A} Y_y(\mathscr{X}; A) = N(A).$$

Therefore all the conditions for Lemma 3.2 hold here, and by that result the conclusion of Theorem 1.2 holds, subject to showing that $\sigma_1 > 0$.

The value of σ_1 is independent of the choice of sequence (A_n) (provided $\liminf(A_n) = \mathbb{Z}^2$) and therefore to show $\sigma_1 > 0$ using (1.3) we are at liberty to choose any sequence $(A_n)_{n \geqslant 1}$. Let $K = \lceil 200\rho \rceil$. Take A_n to be a lattice square of side Kn, and divide $\tilde{A}_n$ into squares of side K, which we shall refer to as *blocks*.

Inside each block S_i let T_i be the annulus of thickness 18ρ, consisting of points at a distance more than 2ρ but less than 20ρ from the boundary of the block. Also let S_i^- be the interior region consisting of points at a distance more than 20ρ from the boundary of the block. Let I_i be the indicator random variable of the event that before there are any arrivals at all in $S_i \setminus T_i$, a sequence of ball centers arrive in T_i in such a way that the corresponding particles are adsorbed without rolling and cause a barrier between the interior region S_i^- and the complement of S_i, by making all points in T_i unavailable.

The probability $P[I_i = 1]$ is very small but not zero, and does not depend on i or n since K is fixed. Let $N_n = \sum_{i=1}^{n^2} I_i$. Then $\mathbb{E}[N_n]/|A_n|$ is a non-zero constant.

Let $\mathscr{F}$ be the σ-field generated by the value of N_n, along with the positions of the accepted particles not lying in the union of the squares $\{S_i^- : I_i = 1\}$. Then

$$\begin{aligned}\operatorname{Var}(H(A_n)) &= \operatorname{Var}(\mathbb{E}[H(A_n) \mid \mathscr{F}]) + \mathbb{E}[\operatorname{Var}(H(A_n \mid \mathscr{F}))] \\ &\geqslant \mathbb{E}[\operatorname{Var}(H(A_n \mid \mathscr{F}))].\end{aligned}$$

Suppose we are given the value of N_n and the configuration of accepted items outside the squares S_i^-, $I_i = 1$. The only remaining variability is from the number of accepted items inside the inner squares S_i^- contributing to N_n.

Let S_i^* be the square consisting of points in S_i at a distance at least 22ρ from S_i, i.e., a slightly smaller square inside S_i^-. We consider two possible lattice configurations inside S_i^*.

Let $\{x_{i,1},..., x_{i,n_1}\}$ be the restriction to S_i^* of a regular triangular lattice with each point distant $(2.02)\,\rho$ from its neighbours (tight packing). Let $\{y_{i,1},..., y_{i,n_2}\}$ be the restriction to S_i^* of a regular triangular lattice with each point distant 3ρ from its neighbours (loose packing). Let E_i be the event that the first n_1 particles in S_i^- to arrive are in the small disks

$D(x_i; 0.01)$, $1 \leqslant i \leqslant n_1$, and are in different disks. Let F_i be the event that the first n_2 particles in S_i^- to arrive are in the disks $D(y_i; 0.01)$, $1 \leqslant i \leqslant n_2$, and are in distinct disks.

Events E_i and F_i have probability bounded away from zero. More particles will be packed into the square S_i^- on event E_i than on event F_i. It follows that there is a constant $c > 0$ such that given $N_n = k$, $\mathrm{Var}(H(A_n) \mid \mathscr{F}) \geqslant ck$. Therefore $\mathrm{Var}(H(A_n)) \geqslant c\mathbb{E}[N_n]$, and this divided by $|A_n|$ is bounded away from zero. Hence $\sigma_1 > 0$ by (1.3). This completes the proof. ∎

Finally we shall prove Theorem 1.4. The aim is to apply Lemma 3.3. Let the point processes ξ_t and ξ (the set of locations of adsorbed particles at time t and at time ∞, respectively) be as defined at (4.2) and (4.3). The family of variables $(\xi(Q_z), z \in \mathbb{Z}^2)$ forms a stationary random field. We need to show rapidly decaying correlations for this random field, and do so via Lemmas 5.4 and 5.5 below. The first of these (but apparently not the second) can be proved using Theorem 4.20 from Chapter I of Liggett,[7] but we take a different approach which is closer to that used already.

For $z \in \mathbb{Z}^d$, let B''_z be the union of all sets B'_y, $y \in B(z; 10\rho + 2)$. Lemmas 4.1 and 4.3 imply an exponentially decaying tail for the diameter of the set B'_0, and hence the distribution of the diameter of B''_z also has an exponentially decaying tail, uniformly in z, i.e.,

$$\limsup_{r \to \infty} \sup_{z \in \mathbb{Z}^d} r^{-1} \log P[B''_z \setminus B(z; r) \neq \varnothing] < 0. \tag{5.6}$$

Lemma 5.4. Given any finite $\Gamma \subset \mathbb{Z}^2$, let $\mathscr{F}_\Gamma$ be the σ-field generated by $(\xi(Q_z), z \in \Gamma)$. There exist positive finite constants K', δ_2 such that if Γ, Γ' are sets in $\mathbb{Z}^2$, both of cardinality at most 4, and the distance between them is $d(\Gamma, \Gamma')$, we have for all events $F \in \mathscr{F}_\Gamma$, $G \in \mathscr{F}_{\Gamma'}$,

$$P[F \cap G] - P[F]\, P[G] \leqslant K' \exp(-\delta_2 d(\Gamma, \Gamma')).$$

Proof. Let $\mathscr{P}'$ be an independent copy of the Poisson process $\mathscr{P}$. Suppose $F \in \mathscr{F}_\Gamma$ and $G \in \mathscr{F}_{\Gamma'}$. Then $F = \{(\xi(Q_z))_{z \in \Gamma} \in R\}$ for some Borel $R \subset \mathbb{R}^\Gamma$, and $G = \{\xi(Q_z)_{z \in \Gamma'} \in R')$ for some Borel $R' \subset \mathbb{R}^{\Gamma'}$.

Let H_Γ be the set of points of $\mathbb{R}^2$ lying closer to Γ than to Γ' and let $H_{\Gamma'} = \mathbb{R}^2 \setminus H_\Gamma$. Let F^* be defined like F but based on points of the Poisson process

$$\mathscr{Q} := (\mathscr{P} \cap (H_\Gamma \times [0, \infty))) \cup (\mathscr{P}' \cap (H_{\Gamma'} \times [0, \infty))).$$

That is, let $\xi^{(1)}$ be defined in the same manner as ξ (Eq. (4.3)) but in terms of the Poisson process $\mathcal{Q}$ instead of $\mathcal{P}$, and let $F^* := \{(\xi^{(1)}(Q_z))_{z \in \Gamma} \in R\}$. Similarly, let G^* be defined like G based on points of

$$\mathcal{Q}' := (\mathcal{P} \cap (H_{\Gamma'} \times [0, \infty))) \cup (\mathcal{P}' \cap (H_\Gamma \times [0, \infty))),$$

that is, let $\xi^{(2)}$ be defined in the same manner as ξ (Eq. (4.3)) but in terms of the Poisson process $\mathcal{Q}'$ instead of $\mathcal{P}$, and let $G^* := \{(\xi^{(2)}(Q_z))_{z \in \Gamma'} \in R'\}$.

Then F^* and G^* are independent (since based on independent Poisson processes $\mathcal{Q}$, $\mathcal{Q}'$), and $P[F^*] = P[F]$, and $P[G^*] = P[G]$. Therefore

$$P[F \cap G] - P[F]\, P[G] = P[F \cap G] - P[F^* \cap G^*]$$

so that

$$|P[F \cap G] - P[F]\, P[G]| \leqslant P[F \triangle F^*] + P[G \triangle G^*].$$

By Lemma 4.4, $F \triangle F^*$ does not occur if $Q_y \subset H_\Gamma$ for all $y \in B''_x$ and all $x \in \Gamma$. Likewise, $G \triangle G^*$ does not occur if $Q_y \subset H_{\Gamma'}$ for all $y \in B''_x$ and all $x \in \Gamma'$. By (5.6), $P[F \triangle F^*]$ and $P[G \triangle G^*]$ both decay exponentially in $d(\Gamma, \Gamma')$, uniformly over finite $\Gamma \subset \mathbb{Z}^2$, $\Gamma' \subset \mathbb{Z}^2$ of cardinality at most 4 and over $F \in \mathcal{F}_\Gamma$, $G \in \mathcal{F}_{\Gamma'}$. ∎

Lemma 5.5. Let $\mathcal{F}_0 = \mathcal{F}_{\{\mathbf{0}\}}$ and let $\mathcal{F}_t$ be the σ-field generated by the variables $\xi(Q_z)$, $z \in \mathbb{Z}^d \setminus B(t)$. Then $\sup\{|P[F \cap G] - P[F]\, P[G]|: F \in \mathcal{F}_0, G \in \mathcal{F}_t\}$ decays exponentially in t.

Proof. Let $\mathcal{P}'$ be an independent copy of the Poisson process $\mathcal{P}$. Suppose $F \in \mathcal{F}_0$ and $G \in \mathcal{F}_t$. Let F^* be defined like F but based on points of

$$(\mathcal{P} \cap (D(\mathbf{0}; t/2) \times [0, \infty))) \cup (\mathcal{P}' \cap (\mathbb{R}^2 \setminus D(\mathbf{0}; t/2)) \times [0, \infty))$$

and let G^* be defined like G but based on points of

$$(\mathcal{P}' \cap (D(\mathbf{0}; t/2) \times [0, \infty))) \cup (\mathcal{P} \cap (\mathbb{R}^2 \setminus D(\mathbf{0}; t/2)) \times [0, \infty)).$$

The precise definition of F^* and G^* is analogous to that used in the preceding proof. Then F^* and G^* are independent, $P[F^*] = P[F]$, and $P[G^*] = P[G]$. Therefore, as in the preceding proof,

$$|P[F \cap G] - P[F]\, P[G]| \leqslant P[F \triangle F^*] + P[G \triangle G^*].$$

By Lemma 4.4, $F \triangle F^*$ does not occur if $B''_\mathbf{0} \subset B(\mathbf{0}; t/4)$. Hence by (5.6), $P[F \triangle F^*]$ decays exponentially in t, uniformly over $F \in \mathcal{F}_0$.

By an extension to the proof of Lemma 5.2, $G \triangle G^*$ does not occur if

$$B(\mathbf{0}; 3t/4) \cap B'_z = \varnothing \qquad \forall z \in \mathscr{A}_{t-2} \cap \mathbb{Z}^d$$

and the probability of this last event decays exponentially in t by (5.6). ▮

Proof of Theorem 1.4. By Lemmas 5.4 and 5.5, the result follows by taking $\Gamma_n = B_n$, in Lemma 3.3, provided we have $\sigma_2 > 0$. An elementary argument shows that with $\psi_z = \xi(Q_z)$,

$$|B_n|^{-1} \operatorname{Var}\left(\sum_{x \in B_n} \psi_x\right) \to \sum_{z \in \mathbb{Z}^2} \operatorname{Cov}(\psi_0, \psi_z) := \sigma_2^2,$$

and the left hand side of this expression can be shown bounded away from zero by a similar argument to that used at the end of the proof of Theorem 1.2. Therefore $\sigma_2 > 0$. ▮

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